京都大学　1972 年　入学試験　理系数学　問題 2

問題

次の式で定められる関数 $F(x)$ に対して，$\lim_{x\to\infty}[F(x)-\log\ x]$ を求めよ．

$$F(x)=\int_0^x \frac{t}{(t+1)(t+3)}\,dt \quad (x>0)$$

解答

$$\begin{aligned}F(x)&=\int_0^x \frac{t}{(t+1)(t+3)}\,dt\\&=\left[\frac{3\log(t+3)}{2}-\frac{\log(t+1)}{2}\right]_0^x\\&=\frac{3\log(x+3)}{2}-\frac{\log(x+1)}{2}-\frac{3\log(0+3)}{2}+\frac{\log(0+1)}{2}\\&=\frac{3\log(x+3)-\log(x+1)-3\log 3}{2}\end{aligned}$$

$$\begin{aligned}&\lim_{x\to\infty}[F(x)-\log\ x]\\&=\lim_{x\to\infty}\left[\frac{3\log(x+3)-\log(x+1)-3\log 3}{2}-\log\ x\right]\\&=\lim_{x\to\infty}\left[\frac{3\log(x+3)-\log(x+1)-2\log\ x-3\log 3}{2}\right]\\&=\lim_{x\to\infty}\left[\frac{\log(x+3)^3/27x^2(x+1)}{2}\right]\\&=\lim_{x\to\infty}\left[\frac{\log\{(x^3+9x^2+27x+27)/27x^2(x+1)\}}{2}\right]\end{aligned}$$

$$\frac{x^3+9x^2+27x+27}{27x^2(x+1)}=\frac{x^3}{27x^2(x+1)}+\frac{9x^2}{27x^2(x+1)}+\frac{27x}{27x^2(x+1)}+\frac{27}{27x^2(x+1)}$$

$$\lim_{x\to\infty}\frac{x^3}{27x^2(x+1)}=\frac{1}{27}$$

$$\lim_{x\to\infty}\frac{9x^2}{27x^2(x+1)}=0$$

$$\lim_{x\to\infty}\frac{27x}{27x^2(x+1)}=0$$

$$\lim_{x\to\infty}\frac{27}{27x^2(x+1)}=0$$

より $\lim_{x\to\infty} \dfrac{x^3+9x^2+27x+27}{27x^2(x+1)} = \dfrac{1}{27}$ よって

$$
\begin{aligned}
&\lim_{x\to\infty}[F(x) - \log\ x] \\
&= \lim_{x\to\infty}\left[\frac{\log\{(x^3+9x^2+27x+27)/27x^2(x+1)\}}{2}\right] \\
&= \frac{\log 1/27}{2} \\
&= -\frac{3\log 3}{2}
\end{aligned}
$$